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Lecture Notes for Massless Spinor and Massive Spinor Triangle Diagrams

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Abstract

These notes present the details of the computation of massless and massive spinor triangle loops for consistent anomalies in gauge theories.

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I. TRIANGLE DIAGRAM FOR MASSLESS LEFT-HANDED SPINOR

We compute the triangle diagrams and study the anomalies [1, 2, 3, 4] for a pure masssless left-handed Weyl spinor in an external gauge field, with the action:

$$S_L = \int d^4x \, \overline{\psi}_L (i\partial\!\!\!/ + V\!\!\!/_L)\psi_L \tag{1}$$

where:

$$V_{L\mu} = B^a_{L\mu} + B^b_{L\mu} + B^c_{L\mu} \tag{2}$$

couples to the current:

$$J_{L\mu} = \overline{\psi}_L \gamma_\mu \psi_L \tag{3}$$

and the components of V_L have the respective masses M^a, M^b, M^c [5].

This is studied in Section I.A, and the resulting consistent anomalies are obtained in I.B. These notes are primarily intended to be a technical memo accompanying [5].

In section II we turn to the case of a finite, and large electron mass, where "large" means in comparison to external momenta and masses. By expanding in inverse powers of m^2 we generate an operator product expansion whose leading term contains the anomaly. We carry out the analysis of the loops in the presence of the full electron mass term, with the couplings

$$\int d^4x \; \left[\overline{\psi}_L (i\partial\!\!\!/ + V\!\!\!/_L)\psi_L + \overline{\psi}_R (i\partial\!\!\!/ + V\!\!\!/_R)\psi_R - m(\overline{\psi}_L \psi_R + h.c.)\right] \tag{4}$$

where we take separate L and R fields:

$$V_{L\mu} = B^{aL}_\mu + B^{bL}_\mu + B^{cL}_\mu \qquad\qquad V_{R\mu} = B^{aR}_\mu + B^{bR}_\mu + B^{cR}_\mu \tag{5}$$

In section 4 we study the anomalies for the left-right symmetric theory which are used in the main text to obtain the full CS term physical process of KK-mode decay. [Note: In the KK-mode case of [5] we have:

$$V_{L\mu} = (-1)^a B^a_\mu + (-1)^b B^b_\mu + (-1)^c B^c_\mu \tag{6}$$

and

$$V_{R\mu} = B^a_\mu + B^b_\mu + B^c_\mu \tag{7}$$

The phase factors $(-1)^n$ correspond to the conventions for KK-modes in ref.[5], and are constructed so that in L-R symmetric theories, fields axial vectors (n odd) couple to $\overline{\psi}\gamma_\mu\gamma^5\psi$

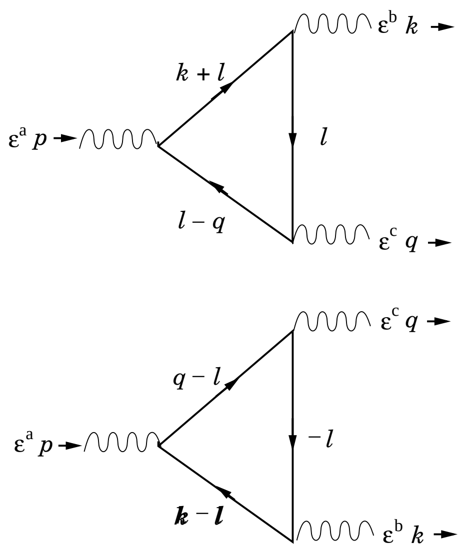

FIG. 1: Bose symmetric triangle diagrams for $B^a(p) \to B^b(k) + B^c(q)$ The external lines are on mass-shell, $p^2 = M_a^2$, $k^2 = M_b^2$ and $q^2 = M_c^2$. The respective polarizations are ϵ^a_μ, ϵ^b_μ and ϵ^c_μ. The internal momentum routing and integration momenta are chosen so that both diagrams have a common denominator.

with a positive sign. For the purely LLL triangle loops these factors are irrelevant, and can be absorbed into the definitions of the fields. In the massive case where the R and L terms interfere, these signs become relevant.]

While we are computing the triangle loop with three distinct external fields, B^a, B^b and B^c, these can be alternatively viewed as distinct momentum components of the single field V. If all three fields were identical (exact Bose invariance) the amplitude would vanish, since it would involve an operator $VVdV$ which is zero. It is the external momentum differences or flavor indices that distinguish these fields and allow non-zero operators such as $[B] \sim B^aB^bdB^c$ and $[B] \sim B^aB^cdB^b$, *etc.*. In the massless Weyl fermion case of interest presently, we compute in a limit $M^a >> M^b \sim M^a \sim 0$. We can view this as an operator product expansion of the triangle diagrams in which the internal lines carrying $p^2 = M_a^2$ are treated as a short-distance expansion.

Both the massless and massive calculations confirm Bardeen's result for the consistent anomalies [3] and provide the necessary terms that maintain overall gauge invariance together with the Chern-Simons term in the process $B^a \to B^b + B^c$, as computed in the text.

A. Massless Weyl Spinor

Our conventions are (Bjorken and Drell, [6]):

$$\epsilon_{0123} = -\epsilon^{0123} = 1 \qquad \gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

$$1 = \epsilon_{0123} = -\epsilon^{0123} \tag{9}$$

With the particular choice of momentum routing in the figure, we have the following expression for the sum of the triangle diagram and its Bose symmetric counterpart, which have a common denominator:

$$\begin{aligned} T &= (-1)(i)^3(i)^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{N_1 + N_2}{D} \\ N_1 &= \mathrm{Tr}[\not{\epsilon}_a L(\not{\ell} - \not{q})\not{\epsilon}_c L(\not{\ell})\not{\epsilon}_b L(\not{\ell} + \not{k})] \\ N_2 &= -\,\mathrm{Tr}[\not{\epsilon}_a L(\not{\ell} + \not{k})\not{\epsilon}_b L(\not{\ell})\not{\epsilon}_c L(\not{\ell} - \not{q})] \\ D &= (\ell + k)^2(\ell^2)(\ell - q)^2 \end{aligned} \tag{10}$$

where:

$$p = k + q; \qquad L = \frac{1}{2}(1 - \gamma^5) \tag{11}$$

and the overall sign contains $\times(i)^3$ (vertices; note that our vector potentials have the opposite sign to the conventions of Bjorken and Drell, hence flipping the vertex rule from $-i\gamma_\mu \to +i\gamma_\mu$)), $\times(i)^3$ (propagators), $\times(-1)$ (Fermi statistics). In N_2 we've factored out an overall minus sign. Note that one must use extreme care to write the given correct cyclic ordering of the factors that make up the numerator, relative to the momentum routing signs. This affects the overall sign of the triangle loop with three gauge vertices (but is has no effect upon the $im\overline{\psi}\gamma^5\psi$ loop computed in the massive case).

[For an alternate cyclic ordering, we have:

$$\begin{aligned} N_1 &= \mathrm{Tr}[\not{\epsilon}_a L(-\not{\ell} - \not{k})\not{\epsilon}_b L(-\not{\ell})\not{\epsilon}_c L(-\not{\ell} + \not{q})] \\ N_2 &= -\,\mathrm{Tr}[\not{\epsilon}_a L(-\not{\ell} + \not{q})\not{\epsilon}_c L(-\not{\ell})\not{\epsilon}_b L(-\not{\ell} - \not{k})] \end{aligned} \tag{12}$$

and the momentum factors appear with minus signs in the numerators. To understand this, consider the a vertex to be located at position x in configuration space, for the first Feynman diagram we have an incoming momentum p, or an $\exp(-ip \cdot x)$ photon wave-function. If we

suppose the b vertex is located at position y. Then the $k+\ell$ line arises from the propagator, $\langle 0|\, T\psi(x)\overline{\psi}(y)\,|0\rangle$, which has a standard Fourier representation of (see, [6], Vol. II, p 185):

$$i\int \frac{d^4h}{(2\pi)^4} e^{-ih\cdot(x-y)} \frac{\not{k}+m}{k^2-m^2} \tag{13}$$

Notice the $\exp(-ih\cdot x)$ factor represents momentum flowing *into* vertex a from vertex b. Likewise, with vertex c located at z, the $\ell-q$ line has an $\exp(+ih'\cdot(x-z))$, factor arising from $\langle 0|\, T\psi(z)\overline{\psi}(x)\,|0\rangle$, represents momentum *outflowing* from vertex a toward vertex c. When we perform the $\int d^4x A_\mu \overline{\psi}\gamma^\mu\psi$, the momentum in the propagators will satisfy $p+h-h'=0$, and we thus choose to define $h=-\ell-k$ and $h'=-\ell+q$.]

We unify the denominator using:

$$\frac{1}{ABC} = 2\int_0^1 dy \int_0^y dz \frac{1}{(Az+B(y-z)+C(1-y))^3} \tag{14}$$

The unified denominator becomes:

$$\frac{1}{D} = 2\int_0^1 dy \int_0^y dz \frac{1}{(\ell^2+2\ell\cdot(zk-(1-y)q)+zk^2+(1-y)q^2)^3} \tag{15}$$

Shifting the loop momentum to a symmetric integration momenta, $\overline{\ell}$:

$$\ell \;=\; \overline{\ell} - zk + (1-y)q \tag{16}$$

the unified denominator becomes:

$$(\overline{\ell}^2 + z(1-z)k^2 + y(1-y)q^2 + 2k\cdot qz(1-y)) \tag{17}$$

We define the following vertex tensors :

$$\begin{aligned}
A &= \epsilon_{\mu\nu\rho\sigma}\epsilon_a^\mu\epsilon_b^\nu\epsilon_c^\rho k^\sigma && \longleftrightarrow && -i\,\langle b,k;c,q|\,\epsilon_{\mu\nu\rho\sigma}B^{a\mu}B^{c\nu}\partial^\rho B^{b\sigma}\,|a,p\rangle \\
B &= \epsilon_{\mu\nu\rho\sigma}\epsilon_a^\mu\epsilon_b^\nu\epsilon_c^\rho q^\sigma && \longleftrightarrow && i\,\langle b,k;c,q|\,\epsilon_{\mu\nu\rho\sigma}B^{a\mu}B^{b\nu}\partial^\rho B^{c\sigma}\,|a,p\rangle \\
C &= \epsilon_{\mu\nu\rho\sigma}\epsilon_a^\mu\epsilon_b^\nu k^\rho q^\sigma && \longleftrightarrow && \frac{1}{2}\,\langle b,k|\,F^a_{\mu\nu}\tilde{F}^{b\,\mu\nu}\,|a,p\rangle \\
D &= \epsilon_{\mu\nu\rho\sigma}\epsilon_a^\mu\epsilon_c^\nu k^\rho q^\sigma && \longleftrightarrow && -\frac{1}{2}\,\langle c,q|\,F^a_{\mu\nu}\tilde{F}^{c\,\mu\nu}\,|a,p\rangle \\
E &= \epsilon_{\mu\nu\rho\sigma}\epsilon_b^\mu\epsilon_c^\nu k^\rho q^\sigma && \longleftrightarrow && \frac{1}{2}\,\langle b,k;c,q|\,F^b_{\mu\nu}\tilde{F}^{c\,\mu\nu}\,|0\rangle
\end{aligned} \tag{18}$$

where we have indicated the corresponding operator matrix elements, ($\langle out|\,\mathcal{O}\,|in\rangle$) and note:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{19}$$

is the standard definition of the dual field strength.

B. Massless Weyl Spinor Triangle Loops

We now compute the triangle loops. Since we are mainly interested in a heavy KK mode decaying to low mass KK-modes, kinematically we have:

$$p = k + q \qquad M_b^2 = k^2 \approx 0 \qquad M_c^2 = q^2 \approx 0 \qquad M_a^2 \approx 2k \cdot q \tag{20}$$

Hence the large M_a^2 limit corresponds to a symmetrical expansion in $k^2/2k \cdot q$ and $q^2/2k \cdot q$. We have, expanding in q^2 and k^2, the following schematic structure:

$$\begin{aligned} T &= \int' \int \frac{d^4\overline{\ell}}{(2\pi)^4} \frac{\alpha(k,q)\overline{\ell}^2 + \beta(k,q)}{(\overline{\ell}^2 + \Delta^2(k^2, k \cdot q, q^2))^3} \\ &= \int' \int \frac{d^4\overline{\ell}}{(2\pi)^4} \left[\frac{(\alpha_0 \overline{\ell}^2 + \beta_0)}{(\overline{\ell}^2 + \Delta_0^2)^3} \left(1 - \frac{3\gamma q^2}{(\overline{\ell}^2 + \Delta_0^2)} - \frac{3\delta k^2}{(\overline{\ell}^2 + \Delta_0^2)} \right) + \frac{\beta_1 q^2 + \beta_2 k^2}{(\overline{\ell}^2 + \Delta_0^2)^3} \right] \end{aligned} \tag{21}$$

where:

$$\begin{aligned} \gamma &= y(1-y), \qquad\qquad \delta = z(1-z), \\ \Delta_0^2 &= 2k \cdot q z(1-y) = M_a^2 z(1-y) \;, \qquad\qquad \int' (\,) = 2\int_0^1 \int_0^y dz\, dy(\,), \end{aligned} \tag{22}$$

and the β_i will be determined below.

For the large mass limit, $|\Delta^2| << m^2$ we define the loop integrals with the usual Wick rotation on the loop energy ℓ_0 and a Euclidean momentum space cut-of Λ^2:

$$\begin{aligned} \int \frac{d^4\ell}{(2\pi)^4} \frac{(1,\ \ell^2)}{(\ell^2 - m^2 + i\epsilon)^3} &= \left[\frac{-i}{16\pi^2} \left(\frac{1}{2m^2} \right), \frac{i}{16\pi^2} \left[\ln\left(\frac{\Lambda^2}{m^2} \right) - \frac{3}{2} \right] \right] \\ \int \frac{d^4\ell}{(2\pi)^4} \frac{(1,\ \ell^2)}{(\ell^2 - m^2 + i\epsilon)^4} &= \left[\frac{i}{16\pi^2} \left(\frac{1}{6(m^2)^2} \right), \frac{-i}{16\pi^2} \left(\frac{1}{3(m^2)^2} \right) \right] \end{aligned} \tag{23}$$

The familiar Wick rotation is a counterclockwise rotation of the contour of the ℓ_0 integral in the complex plane. The rotation is clockwise to avoid the poles at $\pm\sqrt{\vec{l}^2 + m^2} \mp i\epsilon$. The new contour in ℓ_0 runs from $-i\infty$ to $+i\infty$, and we rescale to a Euclidean momentum $\ell_0 = i\overline{\ell}_0$ where $d^4\ell \to i d^4\overline{\ell}$ Since the results are analytic functions of m^2, in the limit $|\Delta_0^2| >> m^2$ we simply replace $m^2 \to -\Delta_0^2$:

We thus have the result:

$$\begin{aligned} T = \frac{i}{16\pi^2} \int' \Bigg[& \alpha_0 \left(\ln\left(\frac{\Lambda^2}{-\Delta_0^2} \right) - \frac{3}{2} + \frac{\gamma q^2}{\Delta_0^2} + \frac{\delta k^2}{\Delta_0^2} \right) \\ & + \beta_0 \left(\frac{1}{2\Delta_0^2} - \frac{\gamma q^2}{2(\Delta_0^2)^2} - \frac{\delta k^2}{2(\Delta_0^2)^2} \right) + \frac{\beta_1 q^2}{2\Delta_0^2} + \frac{\beta_2 k^2}{2\Delta_0^2} \Bigg] \end{aligned} \tag{24}$$

We turn to compute α_0 and the β_i. The numerators are expressed in terms of the shifted momenta and we keep only the γ^5 term in the L projection:

$$
\begin{aligned}
N_1 &= +\frac{1}{2}\,\mathrm{Tr}[\gamma^5 \not\epsilon_a(\bar{\not\ell} - z\not k - y\not q)\not\epsilon_c(\bar{\not\ell} - z\not k + (1-y)\not q)\not\epsilon_b(\bar{\not\ell} + (1-z)\not k + (1-y)\not q)] \\
N_2 &= -\frac{1}{2}\,\mathrm{Tr}[\gamma^5 \not\epsilon_a(\bar{\not\ell} + (1-z)\not k + (1-y)\not q)\not\epsilon_b(\bar{\not\ell} - z\not k + (1-y)\not q)\not\epsilon_c(\bar{\not\ell} - z\not k - y\not q)]
\end{aligned}
\tag{25}
$$

Using the identities:

$$
\begin{aligned}
\not a \not b \not c &= a\cdot b \not c - a\cdot c \not b + b\cdot c \not a + i\epsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho \gamma^5\gamma^\sigma \\
\not a \not b \not c - \not c \not b \not a &= 2i\epsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho \gamma^5\gamma^\sigma \\
\mathrm{Tr}(\not a \not b \not c \not d) &= 4i\epsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma \\
\gamma_\mu\gamma^\nu\gamma^\mu &= -2\gamma^\nu \qquad\qquad \gamma_\mu\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\nu\gamma^\rho
\end{aligned}
\tag{26}
$$

we first compute the $\bar{\ell}^2$ terms, using $\ell_\mu\ell_\nu \to g_{\mu\nu}\ell^2/4$. From the expansion in $\bar{\ell}^2$ of eq.(21), for the numerator terms we have:

$$
N_1 + N_2 \equiv \alpha(k,q)\bar{\ell}^2 + \beta(k,q) \tag{27}
$$

We obtain the leading terms:

$$
\begin{aligned}
\alpha(k,q) &= -\frac{1}{4}\bar{\ell}^2\,\mathrm{Tr}[\gamma^5\not\epsilon_a\not\epsilon_c\not\epsilon_b((1-z)\not k + (1-y)\not q) + \gamma^5\not\epsilon_a(-z\not k - y\not q)\not\epsilon_c\not\epsilon_b \\
&\qquad\qquad +\gamma^5\not\epsilon_a\not\epsilon_b(-z\not k + (1-y)\not q)\not\epsilon_c] \\
&\quad +\frac{1}{4}\bar{\ell}^2\,\mathrm{Tr}[\gamma^5\not\epsilon_a\not\epsilon_b\not\epsilon_c(-z\not k - y\not q) + \gamma^5\not\epsilon_a((1-z)\not k + (1-y)\not q)\not\epsilon_b\not\epsilon_c \\
&\qquad\qquad +\gamma^5\not\epsilon_a\not\epsilon_c(-z\not k + (1-y)\not q)\not\epsilon_b] \\
&= 2i\bar{\ell}^2\big[(1-3z)[\epsilon_{\mu\nu\rho\sigma}\epsilon_a^\mu\epsilon_b^\nu\epsilon_c^\rho k^\sigma] + (2-3y)[\epsilon_{\mu\nu\rho\sigma}\epsilon_a^\mu\epsilon_b^\nu\epsilon_c^\rho q^\sigma]\big]
\end{aligned}
\tag{28}
$$

whence:

$$
\alpha_0 = 2i\big[(1-3z)A + (2-3y)B\big] \tag{29}
$$

We can see that α_0 is Bose symmetric under $b \leftrightarrow \gamma$, $k \leftrightarrow q$.

We also have:

$$
\begin{aligned}
\beta &= -\frac{1}{2}\,\mathrm{Tr}\left[\gamma^5[(1-z)\not k + (1-y)\not q]\right]\not\epsilon_a[z\not k + y\not q]\not\epsilon_c[z\not k - (1-y)\not q]\not\epsilon_b] \\
&\quad +\frac{1}{2}\,\mathrm{Tr}\left[\gamma^5[z\not k + y\not q]\not\epsilon_a[(1-z)\not k + (1-y)\not q]\not\epsilon_b[z\not k - (1-y)\not q]\not\epsilon_c\right]
\end{aligned}
\tag{30}
$$

Note that here we have rearranged the order of terms in $N_2^{(0)}$ in a manner that preserves the manifest Bose symmetry, $c \leftrightarrow b$, $k \leftrightarrow q$. β is readily evaluated:

$$\begin{aligned}
\beta \;=\; & +4i\epsilon_{\mu\nu\rho\sigma}[zk^\mu + yq^\mu]\epsilon_a^\nu[(1-z)k^\rho + (1-y)q^\rho] \times \\
& \quad \big[\epsilon_c^\sigma \epsilon_b \cdot [(zk-(1-y)q] + \epsilon_b^\sigma[(zk-(1-y)q]\cdot\epsilon_c - [zk^\sigma - (1-y)q^\sigma]\epsilon_b\cdot\epsilon_c\big] \\
& -4i\epsilon_{\mu\nu\rho\sigma}\epsilon_b^\mu[zk^\nu - (1-y)q^\nu]\epsilon_c^\rho \times \\
& ([zk^\sigma + yq^\sigma][(1-z)k\cdot\epsilon_a + (1-y)q\cdot\epsilon_a] + [(1-z)k^\sigma + (1-y)q^\sigma][zk\cdot\epsilon_a + yq\cdot\epsilon_a] \\
& \qquad\qquad -\epsilon_a^\sigma[z(1-z)k^2 + (z+y-2yz)k\cdot q + y(1-y)q^2]\big) \\
= \; & 4i[A(z) - B(1-y)]\big(z(1-z)k^2 + (z+y-2yz)k\cdot q + y(1-y)q^2\big) \\
& +4iE\big[k\cdot\epsilon_a(2z - zy - z^2) + q\cdot\epsilon_a(z+y-zy-y^2)\big] \\
& -4i[D][z(z-y)\epsilon_b\cdot k - (z-y)(1-y)\epsilon_b\cdot q] \\
& -4i[C][z(z-y)\epsilon_c\cdot k - (z-y)(1-y)\epsilon_c\cdot q]
\end{aligned} \tag{31}$$

Note that Bose symmetry is manifest in the $[E]$ term since under the replacement, $z \leftrightarrow 1-y$ we have that $(2z - zy - z^2) \leftrightarrow (z + y - zy - y^2)$. Also, $z^2 - zy \leftrightarrow z - y + y^2 - zy$, and $C \leftrightarrow -D$, $\epsilon_b \leftrightarrow \epsilon_c$ makes the symmetry manifest in last two terms.

It is useful to rearrange the β term to reflect the b and c gauge invariance in the $k^2 = q^2 = 0$ limit:

$$\begin{aligned}
\beta \;=\; & 4i[A]\big(2k\cdot qz^2(1-y) + q^2(1-y)(y-z+zy) + k^2z^2(1-z)\big) \\
& -4i[B]\big(2k\cdot qz(1-y)^2 + q^2y(1-y)^2 + k^2z(1-2z+yz)\big) \\
& -4i[Ck\cdot\epsilon_c + Ak\cdot q]z(z-y) \\
& +4i[Cq\cdot\epsilon_c + Aq^2](z-y)(1-y) \\
& -4i[D\epsilon_b\cdot k + Bk^2]z(z-y) \\
& +4i[D\epsilon_b\cdot q + Bk\cdot q](z-y)(1-y) \\
& +4i[E\epsilon_a\cdot k](2z - zy - z^2) + 4i[E\epsilon_a\cdot q](z+y-zy-y^2)
\end{aligned} \tag{32}$$

Here the third through last lines are arranged to vanish if $\epsilon_\mu^b \to k_\mu$ and $\epsilon_\mu^c \to q_\mu$ since $A \to -C$ and $B \to -D$, and $E \to 0$. It can be readily checked that this expression is Bose

symmetric. We can therefore extract the β_i:

$$
\begin{aligned}
\beta_0 \;=\;& 4i[A]\big(2k\cdot q\; z^2(1-y)\big)\\
&-4i[B]\big(2k\cdot q\; z(1-y)^2\big)\\
&-4i[Ck\cdot\epsilon_c + Ak\cdot q]z(z-y)\\
&+4i[Cq\cdot\epsilon_c + Aq^2](z-y)(1-y)\\
&-4i[D\epsilon_b\cdot k + Bk^2]z(z-y)\\
&+4i[D\epsilon_b\cdot q + Bk\cdot q](z-y)(1-y)\\
&+4i[E\epsilon_a\cdot k](2z - zy - z^2) + 4i[E\epsilon_a\cdot q](z + y - zy - y^2)\\
\beta_1 \;=\;& +4i[A](1-y)(z-y+zy) - 4i[B]y(1-y)^2\\
\beta_2 \;=\;& +4i[A]z^2(1-z) - 4i[B]z(1-2y+yz)
\end{aligned}
\tag{33}
$$

We now assemble the result. The superficially log divergent with $k^2 = q^2 = 0$ (the α_0 contribution) yields a finite result:

$$
\begin{aligned}
T_0 \;=\;& \frac{2i}{16\pi^2}\int_0^1 dy\int_0^y dz\;(2i)\big[(1-3z)A + (2-3y)B\big]\left[\ln\left(\frac{\Lambda^2}{-z(1-y)M_a^2}\right) - \frac{3}{2}\right]\\
=\;& -\frac{1}{24\pi^2}[A] + \frac{1}{24\pi^2}[B]
\end{aligned}
\tag{34}
$$

T_0 is Bose symmetric under interchange of the photon and b-KK mode (let $q \leftrightarrow k$, hence $M_b^2 \to 0$, $\epsilon_a \leftrightarrow \epsilon_b$, and note $A \leftrightarrow B$). Note, as a check on the large m^2 case, that if the argument of the log, $\Lambda^2/z(1-y)M_a^2$ is replaced by Λ^2/m^2 then $T_0 = 0$. The result is finite because the integrals:

$$
\int_0^1 dy\int_0^y dz\;\big[(1-3z)A + (2-3y)B\big] \;=\; 0 \tag{35}
$$

This furthermore implies that the imaginary part of the expression is vanishing.

The remaining terms of the triangle diagrams to order q^2, k^2 are:

$$
T_1 \;=\; \frac{2i}{16\pi^2}\int_0^1 dy\int_0^y dz\;\left(\frac{\beta_0}{2\Delta_0^2} - \frac{\beta_0\gamma q^2}{2(\Delta_0^2)^2} + \frac{\alpha_0\gamma q^2}{\Delta_0^2} + \frac{\beta_1 q^2}{2\Delta_0^2}\right) \tag{36}
$$

Combining all of the above terms yields the following full result for the triangle diagrams:

$$
\begin{aligned}
T_0 + T_1 \;=\; & -\frac{1}{12\pi^2}[A] + \frac{1}{12\pi^2}[B] \\
& + \frac{1}{4\pi^2}[Ck \cdot \epsilon_c + Ak \cdot q]\frac{I_b}{M_a^2} \\
& - \frac{1}{4\pi^2}[Cq \cdot \epsilon_c + Aq^2]\frac{I_c}{M_a^2} \\
& + \frac{1}{4\pi^2}[D\epsilon_b \cdot k + Bk^2]\frac{I_b}{M_a^2} \\
& - \frac{1}{4\pi^2}[D\epsilon_b \cdot q + Bk \cdot q]\frac{I_c}{M_a^2} \\
& - \frac{1}{4\pi^2\; M_a^2}[E](\epsilon_a \cdot k\; I_b' + \epsilon_a \cdot q\; I_c') + \mathcal{O}(q^2, k^2). \qquad (37)
\end{aligned}
$$

The integrals I_i and I_i' are infrared divergent in our expansion. The result is manifestly Bose symmetric only if we perform the unification integrals, I_i and I_i' with a Bose symmetric IR cut-off. For a particular choice of small IR cut-offs x_i, the leading log divergent terms are:

$$
\begin{aligned}
I_b &= \int_0^{1-x_b}\int_0^y dz\; dy\; \frac{z(z-y)}{z(1-y)} = \frac{1}{2}\ln(x_b) + k_b \\
I_c &= \int_0^1\int_x^y dz\; dy\; \frac{(1-y)(z-y)}{z(1-y)} = \frac{1}{2}\ln(x_c) + k_c \\
I_b' &= \int_0^{1-x}\int_0^y dz\; dy\; \frac{2z - zy - z^2}{z(1-y)} = -\frac{1}{2}\ln(x_b) + k_b' \\
I_c' &= \int_0^1\int_x^y dz\; dy\; \frac{z + y - zy - y^2}{z(1-y)} = -\frac{1}{2}\ln(x_c) + k_c' \qquad (38)
\end{aligned}
$$

Note that $I_b = I_c$ and $I_b' = I_c'$ if $x_b = x_c$ and Bose symmetry is maintained. The physical cutoffs are of order $x_b \sim M_b^2/M_a^2$ and $x_c \sim M_c^2/M_a^2$, and their exact coefficients are indeterminate in our expansion (hence finite corrections, k and k' to the logs are indeterminate). This can presumably be replaced with a more physical procedure by resumming k^2 and q^2 into the denominators. The logarithmic IR singularities in, *e.g.*, the $q^2 = 0$ limit are presumably cancelled by collinear $\overline{\psi}\psi$ propagation in the $B^c \to B^b + \overline{\psi} + \psi$ process, where $\overline{\psi} + \psi$ rescatter into a photon.

Note a final lemma that is relevant to the anomaly:

$$
I_a + I_b + I_b' + I_c' = 2 \qquad (39)
$$

which is an infra-red "safe" quantity.

C. Massless Weyl Spinor Anomaly

The amplitude T that we have just computed is:

$$T = \langle b, c | \, T ... \, i \int d^4x \;\exp(-ip\cdot x)\epsilon^a_\mu \overline{\psi}\gamma^\mu \psi_L \; ... \, |0\rangle \tag{40}$$

On the other hand, the amplitude we want is the matrix element of the current divergence:

$$\begin{aligned} W &= \langle b, c | \, T ... \int d^4x \;\exp(-ip\cdot x)\partial_\mu \overline{\psi}\gamma^\mu \psi_L \; ... \, |0\rangle \\ &= \langle b, c | \, T ... \int d^4x \;(-\partial_\mu \exp(-ip\cdot x))\overline{\psi}\gamma^\mu \psi_L \; ... \, |0\rangle \\ &= \langle b, c | \, T ... \int d^4x \;\exp(-ip\cdot x) i p_\mu \overline{\psi}\gamma^\mu \psi_L \; ... \, |0\rangle \end{aligned} \tag{41}$$

We thus obtain W from T by the replacement:

$$W = T(\epsilon^a_\mu \rightarrow p_\mu) \tag{42}$$

Under this substitution we have: $[A] \rightarrow -[E]$, $[B] \rightarrow [E]$, $[C] \rightarrow 0$, $[D] \rightarrow 0$, and $\epsilon_a \cdot k \rightarrow k \cdot q$, $\epsilon_a \cdot q \rightarrow k \cdot q$.

We thus obtain:

$$\begin{aligned} T_0 + T_1 \; &\rightarrow \; \frac{1}{12\pi^2}[E] + \frac{1}{12\pi^2}[E] - \frac{1}{8\pi^2}(I_a + I_b + I'_b + I'_c)[E] \\ &= \; -\frac{1}{12\pi^2}[E] \end{aligned} \tag{43}$$

where we use eq.(39). The result is infra red non singular.

Note that we have the operator correspondence $[E] \rightarrow (1/4)F\tilde{F}$. Our result for the anomaly thus corresponds to the operator equation:

$$\partial^\mu \overline{\psi}\gamma_\mu \psi_L = -\frac{1}{48\pi^2} F_{L\mu\nu}\tilde{F}^{\mu\nu}_L \tag{44}$$

which agrees with Bardeen's result for the left-right symmetric anomaly in the case of a massless Weyl spinor [3].

As a further check on the calculation, we can also examine the anomaly in the B^c current, by letting $\epsilon_c \rightarrow -q$ (the minus sign occurs since B^c is outgoing), and we take the c field to be on-shell and massless, *i.e.*. set $q^2 = 0$. Whence $[A] = [C]$ and $[B] = [D] = [E] = 0$:

$$T_0 + T_1 \; \rightarrow \; -\frac{1}{12\pi^2}[C] \tag{45}$$

Using eq.(18) this corresponds to:

$$\partial^\mu \overline{\psi}\gamma_\mu \psi_L = -\frac{1}{48\pi^2} F_{L\mu\nu}\tilde{F}_L^{\mu\nu} \leftrightarrow -\frac{1}{24\pi^2} F_{a\mu\nu}\tilde{F}_b^{\mu\nu} \tag{46}$$

consistent with the a channel result. Likewise, we can check the B^b channel, and verify the same result.

We can furthermore check the off-shell gauge invariance for c identified with a photon and $M_c^2 = 0$. We again set $\epsilon_c \to -q$ and examine the $\mathcal{O}(q^2)$ terms. Whence $[A] = [C]$ and $[B] = [D] = [E] = 0$:

$$\begin{aligned} -\frac{\beta_0 \gamma q^2}{2\Delta_0^4} &\to 2iCq^2\left(\frac{zy(1-y)}{z(1-z)k^2 + 2k\cdot qz(1-y)}\right) \\ \frac{\beta_1 q^2}{2\Delta_0^2} &\to 2iCq^2\left(\frac{(1-y)(z-y-zy)}{z(1-z)k^2 + 2k\cdot qz(1-y)}\right) \\ \frac{\gamma q^2}{\Delta_0^2}\alpha_0 &\to 2iCq^2\left(\frac{(1-3z)y(1-y)}{z(1-z)k^2 + 2k\cdot qz(1-y)}\right) \end{aligned} \tag{47}$$

The sum of these terms in the amplitude is:

$$\begin{aligned} T_1\, q^2 &\to \frac{i}{16\pi^2}(2i[C]q^2)\int_0^1 dy \int_0^y dz \left(\frac{z(1-3z)(1-y)q^2}{z(1-z)k^2 + 2k\cdot qz(1-y)}\right) \\ &= 0 \end{aligned} \tag{48}$$

since the following integral miraculously vanishes:

$$0 = \int_0^1 dy \int_0^y dz \frac{(1-3z)(1-y)}{(1-z)X + (1-y)Y} \tag{49}$$

Thus the $O(q^2)$ terms in the current divergence vanish and off-shell gauge invariance is maintained. The associated operators thus contain internal factors of $\partial_\mu F^{\mu\nu}$. They can presently be set to zero by use of equations of motion. More generally they are easily seen to have associated log IR divergences in the limit of zero electron mass, which are presumably associated with IR singularities in $B^a \to B^b + e^+e^-$ where the massless electron pair becomes indistinguishable from the photon. The IR singularity is cut-off by a nonzero electron mass.

This implies that the only non-gauge invariant part of the amplitude is the anomaly.

II. FINITE ELECTRON MASS

A. Triangle Loops

We now turn to the case of a finite, and large electron mass, where "large" means in comparison to external momenta and masses. By expanding in inverse powers of m^2 we

generate an operator product expansion whose leading term contains the anomaly. We carry out the analysis of the loops in the presence of the full electron mass term, with the couplings

$$\int d^4x \ \left(\overline{\psi}_L(i\partial\!\!\!/ + V\!\!\!/_L)\psi_L + \overline{\psi}_R(i\partial\!\!\!/ + V\!\!\!/_R)\psi_R - m(\overline{\psi}_L\psi_R + h.c.)\right) \tag{50}$$

where we take separate L and R fields, $B_\mu^{a\,L,R}$:

$$V_{L\mu} = B_\mu^{aL} + B_\mu^{bL} + B_\mu^{cL} \qquad\qquad V_{R\mu} = B_\mu^{aR} + B_\mu^{bR} + B_\mu^{cR} \tag{51}$$

[Note that in comparison to the KK-mode normalizations used in [5] we have:

$$B_{L\mu}^n = (-1)^n B_\mu^n \qquad\qquad B_{R\mu}^n = B_\mu^n \tag{52}$$

We will implement this relationship subsequently, but presently we work in the independent and generic V_L, V_R basis.]

We presently adopt an obvious generalized notation for vertices, *e.g.*,

$$\begin{aligned} A^{LRL} &= \epsilon_{\mu\nu\rho\sigma}\epsilon_a^{L\mu}\epsilon_b^{R\nu}\epsilon_c^{L\rho}k^\sigma\ , \qquad\qquad A^{LRR} = \epsilon_{\mu\nu\rho\sigma}\epsilon_a^{L\mu}\epsilon_b^{R\nu}\epsilon_c^{R\rho}k^\sigma \qquad \dots \\ C^{LR} &= \epsilon_{\mu\nu\rho\sigma}\epsilon_a^{L\mu}\epsilon_b^{R\nu}k^\rho q^\sigma \qquad \dots \end{aligned} \tag{53}$$

and so forth.

The LLL (RRR) loops have just been computed, arising from the pure massless ψ_L (ψ_R). In the case of a massive electron the LLL (RRR) loops have the same numerator structure, but the denominator now contains electron mass terms:

$$D \ = \ [(\ell+k) - m^2][(\ell^2)^2 - m^2][(\ell-q)^2 - m^2] \tag{54}$$

This causes all of the previously computed LLL (RRR) terms to become suppressed in the large m^2 limit. For example, the α_0 term previously computed for $m^2 = 0$ now becomes:

$$\begin{aligned} T_0 &= -\frac{1}{4\pi^2}\int_0^1 dy \int_0^y dz\ \Big[(1-3z)A + (2-3y)B\Big]\left[\ln\left(\frac{\Lambda^2}{m^2 - z(1-y)M_a^2}\right) - \frac{3}{2}\right] \\ &\longrightarrow -\frac{M_a^2}{480\pi^2 m^2}\left([A] - [B]\right), \end{aligned} \tag{55}$$

and now vanishes in the large m^2 limit. All of the new terms of interest in the massive electron case arise from the numerator terms containing mass insertions. This represent mixing from ψ_L to the ψ_R, and thus generates new vertices, such as $[A]^{LRL}$, *etc.*.

Let us compute the triangle loops with a single pure left-handed $\epsilon^{aL}_\mu \gamma^\mu L$ vertex, carrying in momentum p, and again noting the the cyclic order in which numerator terms are written:

$$
\begin{aligned}
T_L &= (-1)(i)^3(i)^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{N_1 + N_2}{D} \\
N_1 &= \mathrm{Tr}[\not{\epsilon}_a L(\not{\ell} - \not{q} + m)(\not{\epsilon}^L_c L + \not{\epsilon}^R_c R)(\not{\ell} + m)(\not{\epsilon}^L_b L + \not{\epsilon}^R_b R)(\not{\ell} + \not{k} + m)] \\
N_2 &= -\mathrm{Tr}[\not{\epsilon}_a L(\not{\ell} + \not{k} - m)(\not{\epsilon}^L_b L + \not{\epsilon}^R_b R)(\not{\ell} - m)(\not{\epsilon}^L_c L + \not{\epsilon}^R_c R)(\not{\ell} + \not{q} - m)] \\
D &= [(\ell + k)^2 - m^2][\ell^2 - m^2][(\ell - q)^2 - m^2]
\end{aligned}
\tag{56}
$$

Note the sign flips in the momentum and m terms in N_1 and momenta in N_2, a consequence of having factored out an overall minus sign (note: we could have written a different cyclic order giving the more conventional signs). Upon unifying denominator factors and shifting the loop momentum as before, we obtain the unified denominator:

$$
(\ell^2 + 2\ell \cdot (zk - (1-y)q) + zk^2 + (1-y)q^2 - m^2)^3 \tag{57}
$$

The numerators become:

$$
\begin{aligned}
N_1 &= \mathrm{Tr}[\not{\epsilon}^L_a L(\not{\overline{\ell}} - z\not{k} - y\not{q} + m)(\not{\epsilon}^L_c L + \not{\epsilon}^R_c R) \times \\
&\quad (\not{\overline{\ell}} - z\not{k} + (1-y)\not{q} + m)(\not{\epsilon}^L_b L + \not{\epsilon}^R_b R)(\not{\overline{\ell}} + (1-z)\not{k} + (1-y)\not{q} + m)] \\
N_2 &= -\mathrm{Tr}[\not{\epsilon}^L_a L(\not{\overline{\ell}} + (1-z)\not{k} + (1-y)\not{q} - m)(\not{\epsilon}^L_b L + \not{\epsilon}^R_b R) \times \\
&\quad (\not{\overline{\ell}} - z\not{k} + (1-y)\not{q} - m)(\not{\epsilon}^L_c L + \not{\epsilon}^R_c R)(\not{\overline{\ell}} - z\not{k} - y\not{q} - m)]
\end{aligned}
\tag{58}
$$

Defining the expansion in $\overline{\ell}^2$:

$$
N_1 + N_2 = \alpha_0 \overline{\ell}^2 + \beta + m^2\omega \tag{59}
$$

we see that the α_0 term is just the previously computed LLL term, which now produces a result that vanishes in the large m^2 limit as we have just noted above. Similarly, the β term is as before, entirely composed of LLL terms, and reproduces the previously computed terms in the massless case, but with $1/M_a^2$ now replaced by $1/m^2$. Thus the only new effects of interest are contained in the $m^2\omega$ term.

Expanding the numerators with the shifted loop momentum we have the leading m^2 term:

$$
\begin{aligned}
m^2\omega &= +\frac{m^2}{2}\mathrm{Tr}[\gamma^5 \not{\epsilon}^L_a \not{\epsilon}^R_c \not{\epsilon}^L_b(\not{\overline{\ell}} + (1-z)\not{k} + (1-y)\not{q}) + \gamma^5 \not{\epsilon}^L_a(\not{\overline{\ell}} - z\not{k} - y\not{q})\not{\epsilon}^L_c \not{\epsilon}^R_b + \\
&\qquad \gamma^5 \not{\epsilon}^L_a \not{\epsilon}^R_c(\not{\overline{\ell}} - z\not{k} + (1-y)\not{q})\not{\epsilon}^R_b] \\
&\quad -\frac{m^2}{2}\mathrm{Tr}[\gamma^5 \not{\epsilon}^L_a \not{\epsilon}^R_b \not{\epsilon}^L_c(\not{\overline{\ell}} - z\not{k} - y\not{q}) + \gamma^5 \not{\epsilon}^L_a(\not{\overline{\ell}} + (1-z)\not{k} + (1-y)\not{q})\not{\epsilon}^L_b \not{\epsilon}^R_c + \\
&\qquad \gamma^5 \not{\epsilon}^L_a \not{\epsilon}^R_b(\not{\overline{\ell}} - z\not{k} + (1-y)\not{q})\not{\epsilon}^R_c]
\end{aligned}
\tag{60}
$$

Since $(\overline{\ell})^1$ terms vanish by symmetry, we have:

$$\begin{aligned}
m^2\omega \;=\; & +\frac{m^2}{2}\,\mathrm{Tr}[\gamma^5\not\epsilon^L_a\not\epsilon^R_c\not\epsilon^L_b((1-z)\not k+(1-y)\not q)+\gamma^5\not\epsilon^L_a(-z\not k-y\not q)\not\epsilon^L_c\not\epsilon^R_b+ \\
& \qquad \gamma^5\not\epsilon^L_a\not\epsilon^R_c(-z\not k+(1-y)\not q)\not\epsilon^R_b] \\
& -\frac{m^2}{2}\,\mathrm{Tr}[\gamma^5\not\epsilon^L_a\not\epsilon^R_b\not\epsilon^L_c(-z\not k-y\not q)+\gamma^5\not\epsilon^L_a((1-z)\not k+(1-y)\not q)\not\epsilon^L_b\not\epsilon^R_c+ \\
& \qquad \gamma^5\not\epsilon^L_a\not\epsilon^R_b(-z\not k+(1-y)\not q)\not\epsilon^R_c] \\
=\; & -m^2\,\mathrm{Tr}[\gamma^5\not\epsilon^L_a\not\epsilon^R_b\not\epsilon^L_c(-z\not k-y\not q)]-m^2\,\mathrm{Tr}[\gamma^5\not\epsilon^L_a\not\epsilon^L_b\not\epsilon^R_c((1-z)\not k+(1-y)\not q)] \\
& -m^2\,\mathrm{Tr}[\gamma^5\not\epsilon^L_a\not\epsilon^R_b\not\epsilon^R_c(z\not k-(1-y)\not q)]
\end{aligned} \tag{61}$$

hence:

$$\begin{aligned}
m^2\omega \;=\; & -4im^2\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu L}_a\epsilon^{\nu R}_b\epsilon^{\rho L}_c(-zk^\sigma-yq^\sigma)-4im^2\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu L}_a\epsilon^{\nu L}_b\epsilon^{\rho R}_c((1-z)k^\sigma+(1-y)q^\sigma) \\
& -4im^2\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu L}_a\epsilon^{\nu R}_b\epsilon^{\rho R}_c(zk^\sigma-(1-y)q^\sigma) \\
\equiv\; & -4im^2\big(-z[A]^{LRL}-y[B]^{LRL}\big)-4m^2i\big((1-z)[A]^{LLR}+(1-y)[B]^{LLR}\big) \\
& -4m^2i\big(z[A]^{LRR}-(1-y)[B]^{LRR}\big)
\end{aligned} \tag{62}$$

We thus have the amplitude for the pure $\not\epsilon^a_L L$ vertex:

$$\begin{aligned}
T_L \;=\; & (-4im^2)\times 2\int_0^1 dy\int_0^y dz\;\int\frac{d^4\overline{\ell}}{(2\pi)^4} \\
& \left(\frac{-z[A]^{LRL}-y[B]^{LRL}+(1-z)[A]^{LLR}+(1-y)[B]^{LLR}+z[A]^{LRR}-(1-y)[B]^{LRR}}{(\overline{\ell}^2+z(1-z)k^2+y(1-y)q^2+2k\cdot qz(1-y)-m^2)^3}\right) \\
=\; & -\frac{8m^2}{16\pi^2}\int_0^1 dy\int_0^y dz \\
& \left(\frac{-z[A]^{LRL}-y[B]^{LRL}+(1-z)[A]^{LLR}+(1-y)[B]^{LLR}+z[A]^{LRR}-(1-y)[B]^{LRR}}{2\times(m^2+z(1-z)k^2+y(1-y)q^2+2k\cdot qz(1-y))}\right) \\
=\; & -\frac{1}{4\pi^2}\int_0^1 dy\int_0^y dz \\
& \big(-z[A]^{LRL}-y[B]^{LRL}+(1-z)[A]^{LLR}+(1-y)[B]^{LLR}+z[A]^{LRR}-(1-y)[B]^{LRR}\big)
\end{aligned} \tag{63}$$

This result is negligible in the limit $k^2,\ 2k\cdot q,\ q^2 >> m^2$. However, in the limit of large m^2 we thus have:

$$T_L \;=\; \frac{1}{24\pi^2}([A]^{LRL}+2[B]^{LRL}-2[A]^{LLR}-[B]^{LLR}-[A]^{LRR}+[B]^{LRR}) \tag{64}$$

From this we can easily infer the result for a computation of the triangle loops with a single pure $\epsilon^{aR}_{\mu}\gamma^{\mu}R$ (right-handed) vertex. If we interchange labels $L \leftrightarrow R$, we then flip the overall sign, to obtain:

$$T_R \; = \; -\frac{1}{24\pi^2}([A]^{RLR} + 2[B]^{RLR} - 2[A]^{RRL} - [B]^{RRL} - [A]^{RLL} + [B]^{RLL}) \tag{65}$$

Combining these we have:

$$\begin{aligned} T_L + T_R \; &= \; +\frac{1}{24\pi^2}([A]^{LRL} + 2[B]^{LRL} - 2[A]^{LLR} - [B]^{LLR} - [A]^{LRR} + [B]^{LRR} \\ &\quad -[A]^{RLR} - 2[B]^{RLR} + 2[A]^{RRL} + [B]^{RRL} + [A]^{RLL} - [B]^{RLL}) \end{aligned} \tag{66}$$

[Now consider the application to KK-modes following [5]. For KK-mode B^n_μ we have an x^5 wave-function parity of $(-1)^n$, and $B^n_{\mu L} = (-1)^n B^n_{\mu R} = B^n_\mu$. The KK-modes are normalized so that an axial vector (odd n) couples to $\overline{\psi}\gamma_\mu\gamma^5\psi$ with positive sign. Thus, we can write:

$$\begin{aligned} T_L + T_R \; &= \; \frac{1}{24\pi^2}((-1)^{a+c}([A] + 2[B]) - (-1)^{a+b}(2[A] + [B]) - (-1)^{a}([A] - [B]) \\ &\quad -(-1)^{b}([A] + 2[B]) + (-1)^{c}(2[A] + [B]) + (-1)^{b+c}([A] - [B]) \end{aligned} \tag{67}$$

This can be put into a compact final expression:

$$T_L + T_R \; = \; \frac{1}{12\pi^2}(f_{abc}[A] + g_{abc}[B]) \tag{68}$$

where:

$$\begin{aligned} f_{abc} \; &= \; \frac{1}{2}((-1)^{a+c} - 2(-1)^{a+b} - (-1)^{a} - (-1)^{b} + 2(-1)^{c} + (-1)^{b+c}) \\ g_{abc} \; &= \; \frac{1}{2}(2(-1)^{a+c} - (-1)^{a+b} + (-1)^{a} - 2(-1)^{b} + (-1)^{c} - (-1)^{b+c}). \end{aligned} \tag{69}$$

Note that if $a+b+c$ is even, then $f = g = 0$, which is the condition that a transition cannot occur! But, of course, the *condition that a transition can occur* is $a + b + c$ odd. When $a + b + c$ is odd, we can therefore write:

$$\begin{aligned} f_{abc} \; &= \; -(-1)^{a} - (-1)^{b} + 2(-1)^{c} \\ g_{abc} \; &= \; (-1)^{a} - 2(-1)^{b} + (-1)^{c} \end{aligned} \tag{70}$$

Under $b \leftrightarrow c$ we have $A \leftrightarrow -B$ and thus $g_{abc} \leftrightarrow -f_{acb}$, which checks. Under the Bose exchange $a \leftrightarrow b$ we have $B \to -B$ and $A \to A+B$ (since the k in the A vertex now becomes

$-k-q$ with the sign flip since a is incoming momentum $k+q$ and b is outgoing momentum k). Thus the vertex becomes:

$$T_L + T_R \to \frac{1}{12\pi^2}(f_{bac}[A] + (f_{bac} - g_{bac})[B]) \tag{71}$$

and we immediately verify that $f_{bac} = f_{abc}$ and $f_{bac} - g_{bac} = g_{abc}$. Thus the amplitude is seen to be fully Bose symmetric (we leave the verification of $a \leftrightarrow c$ Bose symmetry to the reader).

The vertex calculation can be represented by an operator of the form:

$$\mathcal{O} = -\frac{1}{12\pi^2}\epsilon^{\mu\nu\rho\sigma}\sum_{nmk} a_{nmk} B^n_\mu B^m_\nu \partial_\rho B^k_\sigma \tag{72}$$

where:

$$a_{nmk} = \frac{1}{2}[1 - (-1)^{n+m+k}](-1)^{m+k} \tag{73}$$

For the process $a \to b + c$ the matrix element of $\mathcal{O}$ takes the form (we've multiplied by $+i$ from e^{iS}):

$$i\langle a|\,\mathcal{O}\,|b,c\rangle \;=\; \frac{1}{12\pi^2}\left[(-a_{abc} + a_{bac} + a_{bca} - a_{cba})[B] + (a_{acb} - a_{cab} + a_{bca} - a_{cba})[A]\right] \tag{74}$$

and we see that (for $a+b+c$ odd):

$$\begin{aligned} -a_{abc} + a_{bac} + a_{bca} - a_{cba} &= g_{abc} \\ a_{acb} - a_{cab} + a_{bca} - a_{cba} &= f_{abc} \end{aligned} \tag{75}$$

B. Massive Left-Right Symmetric Anomaly

The current divergence, $\partial_\mu\overline{\psi}\gamma_\mu\psi_L$, is obtained by the replacement $\epsilon_\mu \to p_\mu$ in T_L. We thus have that $A \to -E$ and $B \to E$:

$$\langle 0|\,\partial_\mu\overline{\psi}\gamma_\mu\psi_L\,|b,c\rangle \;=\; \frac{1}{24\pi^2}([E]^{RL} + [E]^{LR} + 2[E]^{RR}) \tag{76}$$

Likewise:

$$\langle 0|\,\partial_\mu\overline{\psi}\gamma_\mu\psi_R\,|b,c\rangle \;=\; -\frac{1}{24\pi^2}([E]^{LR} + [E]^{RL} + 2[E]^{LL}) \tag{77}$$

This result is the current divergence including the loop numerator mass insertions:

$$\partial_\mu \overline{\psi}\gamma_\mu\psi_L \;=\; \frac{1}{48\pi^2}(F^L_{\mu\nu}\tilde{F}^R_{\mu\nu} + F^R_{\mu\nu}\tilde{F}^R_{\mu\nu}) \tag{78}$$

$$\partial_\mu \overline{\psi}\gamma_\mu\psi_R \;=\; -\frac{1}{48\pi^2}(F^L_{\mu\nu}\tilde{F}^R_{\mu\nu} + F^L_{\mu\nu}\tilde{F}^L_{\mu\nu}) \tag{79}$$

We emphasize that this result is *not the anomaly*. To extract the anomaly, we note that the equations of motion yield the divergences of the spinor currents:

$$\partial^\mu \overline{\psi}\gamma_\mu\psi_L = -im(\overline{\psi}_L\psi_R - \overline{\psi}_R\psi_L) + \text{anomaly} \tag{80}$$

$$\partial^\mu \overline{\psi}\gamma_\mu\psi_R = -im(\overline{\psi}_R\psi_L - \overline{\psi}_L\psi_R) + \text{anomaly} \tag{81}$$

We thus need to subtract the vacuum to 2-gauge field matrix element of the mass term, which is the operator $-im\overline{\psi}\gamma^5\psi$, to obtain the anomaly. The mass term has a similar structure to the triangle diagrams, and we define:

$$\begin{aligned}
M^5 &= (-1)(i)^2(i)^3 \int' \int \frac{d^4\ell}{(2\pi)^4}\frac{N_1+N_2}{D} \\
N_1 &= (-i)(-im)\,\mathrm{Tr}[\gamma^5(\not\ell - \not q + m)(\not\epsilon^L_c L + \not\epsilon^R_c R)(\not\ell + m)(\not\epsilon^L_b L + \not\epsilon^R_b R)(\not\ell + \not k + m)] \\
N_2 &= (+i)(-im)\,\mathrm{Tr}[\gamma^5(\not\ell + \not k - m)(\not\epsilon^L_b L + \not\epsilon^R_b R)(\not\ell - m)(\not\epsilon^L_c L + \not\epsilon^R_c R)(\not\ell - \not q - m)] \\
D &= (\ell^2 + 2\ell\cdot(zk - (1-y)q) + zk^2 + (1-y)q^2 - m^2)^3
\end{aligned} \tag{82}$$

The numerators become:

$$\begin{aligned}
N_1 &= -m\,\mathrm{Tr}[\gamma^5(\not\overline{\ell} - z\not k - y\not q + m)(\not\epsilon^L_c L + \not\epsilon^R_c R)\times \\
&\quad (\not\overline{\ell} - z\not k + (1-y)\not q + m)(\not\epsilon^L_b L + \not\epsilon^R_b R)(\not\overline{\ell} + (1-z)\not k + (1-y)\not q + m)] \\
N_2 &= m\,\mathrm{Tr}[\gamma^5(\not\overline{\ell} + (1-z)\not k + (1-y)\not q - m)(\not\epsilon^L_b L + \not\epsilon^R_b R)\times \\
&\quad (\not\overline{\ell} - z\not k + (1-y)\not q - m)(\not\epsilon^L_c L + \not\epsilon^R_c R)(\not\overline{\ell} - z\not k - y\not q - m)]
\end{aligned} \tag{83}$$

Defining the expansion in $\overline{\ell}^2$:

$$N_1 + N_2 = m^2 N^{(2)}\overline{\ell}^2 + m^2\omega'\overline{\ell}^0 \tag{84}$$

we see that $N^{(2)} = 0$ since it reduces to $\mathrm{Tr}(\gamma^5\not a\not b) = 0$. Thus, the ω' terms are:

$$\begin{aligned}
m^2\omega' &= -m\,\mathrm{Tr}[\gamma^5(-z\not k - y\not q + m)(\not\epsilon^L_c L + \not\epsilon^R_c R)\times \\
&\quad (-z\not k + (1-y)\not q + m)(\not\epsilon^L_b L + \not\epsilon^R_b R)((1-z)\not k + (1-y)\not q + m)] \\
&\quad +m\,\mathrm{Tr}[\gamma^5((1-z)\not k + (1-y)\not q - m)(\not\epsilon^L_b L + \not\epsilon^R_b R)\times \\
&\quad (-z\not k + (1-y)\not q - m)(\not\epsilon^L_c L + \not\epsilon^R_c R)(-z\not k - y\not q - m)]
\end{aligned} \tag{85}$$

We can write:

$$\begin{aligned} m^2\omega' &= -m\,\mathrm{Tr}[(\not{B}+m)\gamma^5(\not{A}+m)(\not{\epsilon}_c)(\not{C}+m)(\not{\epsilon}_b)] \\ & \quad +m\,\mathrm{Tr}[(\not{A}-m)\gamma^5(\not{B}-m)(\not{\epsilon}_b)(\not{C}-m)(\not{\epsilon}_c)] \end{aligned} \tag{86}$$

where:

$$\begin{aligned} \not{A} &= -z\not{k}-y\not{q} \qquad \not{B} = (1-z)\not{k}+(1-y)\not{q} \qquad \not{C} = -z\not{k}+(1-y)\not{q} \\ \not{\epsilon}_b &= \not{\epsilon}_b^L L+\not{\epsilon}_b^R R \qquad \not{\epsilon}_c = \not{\epsilon}_c^L L+\not{\epsilon}_c^R R \end{aligned} \tag{87}$$

$$\begin{aligned} m^2\omega' &= -m^2\,\mathrm{Tr}[(\not{B}\gamma^5+\gamma^5\not{A})(\not{\epsilon}_c\not{C}\not{\epsilon}_b)+\not{B}\gamma^5\not{A}\not{\epsilon}_c\not{\epsilon}_b] \\ & \quad -m^2\,\mathrm{Tr}[(\not{A}\gamma^5+\gamma^5\not{B})(\not{\epsilon}_b\not{C}\not{\epsilon}_c)+\not{A}\gamma^5\not{B}\not{\epsilon}_b\not{\epsilon}_c] \\ &= -m^2\,\mathrm{Tr}[(\gamma^5(\not{A}-\not{B})[\not{\epsilon}_c\not{C}\not{\epsilon}_b-\not{\epsilon}_b\not{C}\not{\epsilon}_c]-\gamma^5\not{A}\not{B}(\not{\epsilon}_b\not{\epsilon}_c-\not{\epsilon}_c\not{\epsilon}_b)] \end{aligned} \tag{88}$$

Compute, for example, the $\epsilon_b^L\epsilon_c^L$ terms:

$$\begin{aligned} &= -m^2\,\mathrm{Tr}[\gamma^5\not{\epsilon}_b^L\not{\epsilon}_c^L(\not{A}-\not{B})\not{C}] \\ &= -m^2\,\mathrm{Tr}[\gamma^5\not{\epsilon}_b^L\not{\epsilon}_c^L(-\not{k}-\not{q})(-z\not{k}+(1-y)\not{q})] \\ &= 4im^2[E](1+z-y) \end{aligned} \tag{89}$$

Full result:

$$m^2\omega' = 4im^2[(E^{LL}+E^{RR})(1+z-y)+(E^{LR}+E^{RL})(y-z)] \tag{90}$$

and:

$$\begin{aligned} M^5 &= \frac{-1}{16\pi^2}\left(\frac{1}{2m^2}\right)2\times(-4m^2)[\frac{1}{3}(E^{LL}+E^{RR})+\frac{1}{6}(E^{LR}+E^{RL})] \\ &= \frac{1}{24\pi^2}[2(E^{LL}+E^{RR})+(E^{LR}+E^{RL})] = \langle 0|-im\overline{\psi}\gamma^5\psi\,|b,c\rangle \end{aligned} \tag{91}$$

or:

$$im\overline{\psi}\gamma^5\psi \to -\frac{1}{48\pi^2}[F_L\tilde{F}_L+F_R\tilde{F}_R+F_L\tilde{F}_R] \tag{92}$$

Forming the difference of the current divergence with $-im\overline{\psi}\gamma^5\psi$ we have:

$$\begin{aligned} &\partial_\mu\overline{\psi}\gamma_\mu\psi_L+im(\overline{\psi}_L\psi_R-\overline{\psi}_R\psi_L) = \\ &\frac{1}{48\pi^2}(F_L\tilde{F}_R+F_R\tilde{F}_R)-\frac{1}{48\pi^2}[F_L\tilde{F}_L+F_R\tilde{F}_R+F_L\tilde{F}_R] \\ &= -\frac{1}{48\pi^2}F_L\tilde{F}_L \end{aligned} \tag{93}$$

Likewise:

$$\begin{aligned}
&\partial_\mu \overline{\psi}\gamma_\mu\psi_R + im(\overline{\psi}_R\psi_L - \overline{\psi}_L\psi_R) = \\
&\quad -\frac{1}{48\pi^2}(F_R\tilde{F}_L + F_L\tilde{F}_L) + \frac{1}{48\pi^2}[F_L\tilde{F}_L + F_R\tilde{F}_R + F_L\tilde{F}_R] \\
&= \frac{1}{48\pi^2}F_R\tilde{F}_R
\end{aligned} \tag{94}$$

III. THE COVARIANT ANOMALY

The consistent anomalies for the vector and axial vector currents can be written in the form:

$$\begin{aligned}
\partial_\mu J^\mu &= \frac{1}{12\pi^2}F_V^{\mu\nu}\tilde{F}_{A\mu\nu} \\
\partial_\mu J^{5\mu} &= \frac{1}{24\pi^2}[F_V^{\mu\nu}\tilde{F}_{V\mu\nu} + F_A^{\mu\nu}\tilde{F}_{A\mu\nu}]
\end{aligned} \tag{95}$$

where $V_L = V - A$ and $V_R = V + A$ and:

$$J = J_L + J_R\ , \qquad\qquad J^5 = J_R - J_L\ . \tag{96}$$

A unique term can be added to the action of the form:

$$S' = \frac{1}{6\pi^2}\int d^4x\ \epsilon_{\mu\nu\rho\sigma}A^\mu V^\nu \partial^\rho V^\sigma\ . \tag{97}$$

S' has even parity and is nonvanishing. Upon variation wrt V or A, it adds corrections to the vector and axial currents:

$$\begin{aligned}
\frac{\delta S'}{\delta V_\mu} &= \delta J^\mu = -\frac{1}{3\pi^2}\epsilon_{\mu\nu\rho\sigma}A^\nu\partial^\rho V^\sigma + \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}V^\nu\partial^\rho A^\sigma \\
\frac{\delta S'}{\delta A_\mu} &= \delta J^{5\mu} = \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}V^\nu\partial^\rho V^\sigma
\end{aligned} \tag{98}$$

We see that:

$$\begin{aligned}
\partial_\mu(\delta J^\mu) &= -\frac{1}{12\pi^2}\epsilon_{\mu\nu\rho\sigma}F_V^{\mu\nu}F_A^{\rho\sigma} \\
\partial_\mu(\delta J^{5\mu}) &= \frac{1}{12\pi^2}\epsilon_{\mu\nu\rho\sigma}F_V^{\mu\nu}F_V^{\rho\sigma}
\end{aligned} \tag{99}$$

The full currents, $\tilde{J} = J + \delta J$, now satisfy:

$$\partial_\mu\tilde{J}^\mu = 0\ , \qquad\qquad \partial_\mu\tilde{J}^{5\mu} = \frac{1}{8\pi^2}\left(F_V^{\mu\nu}\widetilde{F}_{V\mu\nu} + \frac{1}{3}F_A^{\mu\nu}\widetilde{F}_{A\mu\nu}\right)\ . \tag{100}$$

This is called the "*covariant*" form of the anomaly. The theory is now invariant and operators transform covariantly with respect to the *vector* gauge symmetry.

IV. SUMMARY

Pseudoscalar Mass Term:

$$im\overline{\psi}\gamma^5\psi \to -\frac{1}{48\pi^2}[F_{L\mu\nu}\tilde{F}_L^{\mu\nu} + F_{R\mu\nu}\tilde{F}_R^{\mu\nu} + F_{L\mu\nu}\tilde{F}_R^{\mu\nu}] \tag{101}$$

Consistent Anomalies:

(1) Pure Massless Weyl Spinors ($p_i \cdot p_j >> m^2$):

$$\begin{aligned} \partial^\mu\overline{\psi}\gamma_\mu\psi_L &= -\frac{1}{48\pi^2}F_{L\mu\nu}\tilde{F}_L^{\mu\nu} \\ \partial^\mu\overline{\psi}\gamma_\mu\psi_R &= \frac{1}{48\pi^2}F_{R\mu\nu}\tilde{F}_R^{\mu\nu} \end{aligned} \tag{102}$$

(2) Heavy Massive Weyl Spinors ($p_i \cdot p_j << m^2$):

$$\begin{aligned} \partial^\mu\overline{\psi}\gamma_\mu\psi_L + im(\overline{\psi}_L\psi_R - \overline{\psi}_R\psi_L) &= -\frac{1}{48\pi^2}F_{L\mu\nu}\tilde{F}_L^{\mu\nu} \\ \partial^\mu\overline{\psi}_R\gamma_\mu\psi_R + im(\overline{\psi}_R\psi_L - \overline{\psi}_L\psi_R) &= \frac{1}{48\pi^2}F_{R\mu\nu}\tilde{F}_R^{\mu\nu} \end{aligned} \tag{103}$$

(3) Heavy Massive Weyl Spinors ($p_i \cdot p_j << m^2$):

$$\begin{aligned} \partial^\mu\overline{\psi}\gamma_\mu\psi_L &= \frac{1}{48\pi^2}(F_{L\mu\nu}\tilde{F}_R^{\mu\nu} + F_{R\mu\nu}\tilde{F}_R^{\mu\nu}) \\ \partial^\mu\overline{\psi}\gamma_\mu\psi_R &= -\frac{1}{48\pi^2}(F_{L\mu\nu}\tilde{F}_R^{\mu\nu} + F_{L\mu\nu}\tilde{F}_L^{\mu\nu}) \end{aligned} \tag{104}$$

Consistent $L = V - A$ and $R = V + A$ Forms:

(1) Pure Massless Weyl Spinors ($p_i \cdot p_j >> m^2$):

$$\begin{aligned} \partial^\mu\overline{\psi}\gamma_\mu\psi &= \frac{1}{12\pi^2}F_{V\mu\nu}\tilde{F}_A^{\mu\nu} \\ \partial^\mu\overline{\psi}\gamma_\mu\gamma^5\psi &= \frac{1}{24\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu} + F_{A\mu\nu}\tilde{F}_A^{\mu\nu}) \end{aligned} \tag{105}$$

(2) Heavy Massive Weyl Spinors ($p_i \cdot p_j << m^2$):

$$\begin{aligned} \partial^\mu\overline{\psi}\gamma_\mu\psi &= \frac{1}{12\pi^2}F_{V\mu\nu}\tilde{F}_A^{\mu\nu} \\ \partial^\mu\overline{\psi}\gamma_\mu\gamma^5\psi - 2im\overline{\psi}\gamma^5\psi &= \frac{1}{24\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu} + F_{A\mu\nu}\tilde{F}_A^{\mu\nu}) \end{aligned} \tag{106}$$

(3) Heavy Massive Weyl Spinors ($p_i \cdot p_j << m^2$):

$$\begin{aligned} \partial^\mu \overline{\psi}\gamma_\mu\psi &= \frac{1}{12\pi^2} F_{V\mu\nu}\tilde{F}_A^{\mu\nu} \\ \partial^\mu \overline{\psi}\gamma_\mu\gamma^5\psi &= -\frac{1}{12\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu}) \end{aligned} \tag{107}$$

Covariant Forms:

Add a term to the lagrangian of the form $(1/6\pi^2)\epsilon_{\mu\nu\rho\sigma}A^\mu V^\nu \partial^\rho V^\sigma$. The currents are now modified to $\tilde{J} = J + \delta J$ and $\tilde{J}^5 = J^5 + \delta J^5$ as described in the text.

(1) Pure Massless Weyl Spinors ($p_i \cdot p_j >> m^2$):

$$\begin{aligned} \partial^\mu \tilde{J}_\mu &= 0 \\ \partial^\mu \tilde{J}^5_\mu &= \frac{1}{8\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu} + \frac{1}{3}F_{A\mu\nu}\tilde{F}_A^{\mu\nu}) \end{aligned} \tag{108}$$

(2) Heavy Massive Weyl Spinors ($p_i \cdot p_j << m^2$):

$$\begin{aligned} \partial^\mu \tilde{J}_\mu &= 0 \\ \partial^\mu \tilde{J}^5_\mu - 2im\overline{\psi}\gamma^5\psi &= \frac{1}{8\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu} + \frac{1}{3}F_{A\mu\nu}\tilde{F}_A^{\mu\nu}) \end{aligned} \tag{109}$$

(3) Heavy Massive Weyl Spinors ($p_i \cdot p_j << m^2$):

$$\begin{aligned} \partial^\mu \tilde{J}_\mu &= 0 \\ \partial^\mu \tilde{J}^5_\mu &= 0 \end{aligned} \tag{110}$$

The latter case is completely summarized by the fact that, for KK-modes, the three-gauge boson amplitude is described by the operator:

$$\mathcal{O} = -\frac{1}{12\pi^2}\epsilon^{\mu\nu\rho\sigma}\sum_{nmk} a_{nmk} B^n_\mu B^m_\nu \partial_\rho B^k_\sigma \tag{111}$$

where:

$$a_{nmk} = \frac{1}{2}[1 - (-1)^{n+m+k}](-1)^{m+k} \tag{112}$$

This operator is equivalent to $(-1/6\pi^2)\epsilon_{\mu\nu\rho\sigma}A^\mu V^\nu \partial^\rho V^\sigma$ when we truncate on the first two KK-modes, and identify $B^0 = V$ and $B^1 = A$. Adding the $(1/6\pi^2)\epsilon_{\mu\nu\rho\sigma}A^\mu V^\nu \partial^\rho V^\sigma$ term cancels this quantity, completely cancels the 3-gauge boson triangle diagrams, and the resulting currents then have vanishing divergences.

[1] J. S. Bell and R. Jackiw, Nuovo Cim. A **60**, 47 (1969).

[2] S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[3] W. A. Bardeen, Phys. Rev. **184**, 1848 (1969).

[4] See also: I. S. Gerstein and R. Jackiw, Phys. Rev. **181**, 1955 (1969).

[5] C. T. Hill, "Anomalies, Chern Simons terms, and Chiral delocalization in Extra Dimensions," hep-th/0601154.

[6] J. D. Bjorken and S. D. Drell, "Relativistic Quantum Field Theory," McGraw-Hill (1964).